Income distribution and cyclical dynamics in a supermultiplier model

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Abstract

Recent extensions of the Sraffian Supermultiplier model (SSM) [Serrano (1995) and Freitas and Serrano (2015)] and studies, including those by Summa et al. (2023) and Araujo et al. (2023) have explored the model's ability to account for cyclical fluctuations showing that the SSM model also provides valuable insights into the cyclical dynamics of capacity utilization and the share of the investment in the output. On another front, Morlin and Pariboni (2023) drawing on the seminal contribution by Rowthorn (1977), have endogenized income distribution in the SSM through a conflict inflation channel whereby income distribution dynamics are influenced by labour market conditions. In the present paper, we merge these two lines of investigation by analyzing the conditions for the emergence of endogenous cycles in the extended SSM advanced by Morlin and Pariboni. As in the analysis carried out by Araujo et al., we have adopted the Hopf bifurcation theorem to check if the conditions for the emergence of cyclical behaviour are present in the Morlin and Pariboni model. Through a formal analysis of the dynamic stability of the model equilibrium path, we identify the necessary conditions for a Hopf bifurcation for each parameter. Our findings show that persistent endogenous cycles are consistent with the extended SSM and can help us to understand the dynamics of inflation, income distribution and growth.

1. Introduction

Cyclical patterns are a key stylized fact of advanced capitalist economies [see Goodwin (1950, 1951) and Kalecki (1954)]. The ability to generate self-sustained oscillations through non-linear differential equations, effectively replicating business cycles, has been explored under different perspectives within demand-led macroeconomic models. Yet, it remains an open area of inquiry. Recently, the debate on endogenous cycles has extended to the autonomous demand-led growth models, e.g. the Sraffian Supermultiplier model (SSM) (Serrano, 1995; Freitas and Serrano, 2015). Critics of the SSM have questioned its ability to generate persistent cycles, thus challenging its effectiveness in explaining economic growth and the overall dynamics of aggregate demand (Nikiforos et al., 2023). In light of these objections, Araujo et al. (2023), have expanded the SSM's scope to account for cyclical fluctuations, demonstrating that the model can provide valuable insights into the cyclical dynamics of capacity utilization and the investment share in output. An alternative response was provided by Summa et al. (2023) who emphasized the cyclical dynamics of autonomous demand itself as the explanation of economic fluctuations.

Notwithstanding recent contributions, the cyclical dynamics of employment and income distribution remain largely unexplored in the SSM literature, which represents a central theme in the demand-driven literature on economic fluctuations. We address this issue by analyzing the extended SSM proposed by Morlin and Pariboni (2024), which endogenizes the dynamics of income distribution through a conflict inflation mechanism. Building on the analysis of Araujo et al. (2023), we apply the Hopf bifurcation theorem to determine whether cyclical behaviour can emerge within the Morlin-Pariboni model. Through a formal analysis of the dynamic stability of the model's equilibrium, we take an important step forward in demonstrating the conditions under which the model admits limit cycles, suggesting the potential for sustained oscillatory behaviour. To achieve this, we employ the stability criterion developed by Asada and Yoshida (2003), which provides a robust framework for analyzing the qualitative properties of four-dimensional dynamic economic models. This criterion allows us to explore the system's local dynamics around the equilibrium point and to identify bifurcations by focusing on the relationships between the model's parameters and the structure of the Jacobian matrix, particularly in the presence of complex eigenvalues. By applying this method, we find that the interaction between conflict inflation, unemployment, and autonomous demand can generate limit cycles, leading to endogenous and persisting fluctuations in key variables such as wage share and capacity utilization.

We contribute to the literature on demand-driven business cycles by showing how persistent fluctuations can emerge in an autonomous demand-led growth model with endogenous distribution (as advanced by Morlin and Pariboni, 2024). We refine the dynamic properties of the SSM with endogenous distribution and conflict inflation. By applying the Asada and Yoshida criterion, we offer a deeper understanding of how conflict inflation can interact with growth dynamics to produce sustained cycles, thereby enriching the theoretical foundations of demand-led growth models. Our findings demonstrate that persistent endogenous cycles are consistent with the extended SSM without relying upon any *ad* hoc assumption of non-linearity. In this way, we offer insights into the interconnected dynamics of inflation, income distribution, and growth.

The paper is structured as follows. Section 2 reviews the relevant literature. Section 3 revisits the dynamics of the model and conducts a stability analysis. Section 4 introduces the Asada and Yoshida criterion for cyclical behaviour. Finally, Section 5 concludes with a discussion of the implications and potential avenues for future research.

2. Distribution, autonomous demand, and cycles

The Sraffian Supermultiplier model (SSM) is a contribution to demand-led growth theory (Serrano, 1995; Freitas and Serrano, 2015) that highlights the role of autonomous components of aggregate demand that do not generate productive capacity as the drivers of economic growth. The SSM ensures an endogenous convergence of the degree of capacity utilization towards normal capacity without necessarily triggering Harrodian instability, which is prevented by a sufficiently slow adjustment of the investment rate to the use of installed capacity (Freitas and Serrano, 2015; see also Haluska et al., 2021). In the SSM steady state – *i.e.*, the fully adjusted position -, the growth rate of non-capacity generating autonomous expenditures determines the growth rate of aggregate demand, productive capacity, and output in the long run (Serrano, 1995).

The SSM does not rely on endogenous adjustments in income distribution nor does it require any functional relation between growth and distribution to achieve its fully adjusted position. Sraffian and neo-Kaleckian growth models typically treat income distribution as an exogenous variable, reflecting a conceptual separation between two logical domains: the determination of distributive variables and income shares, and the processes governing the level and growth of output. Treating income distribution as exogenous in these models therefore does not imply that it is unaffected by growth. Rather, this approach allows for a clearer analysis of how distribution and growth interact.

The interplay between autonomous demand-led growth and income distribution was explored by different authors. Notably, Nah and Lavoie (2019) and Brochier (2020) combined the SSM with conflict inflation, by including a labour market effect on wage claims, and endogenous labour supply. However, their approach uses the rate of change in employment, thereby resulting in the distributive effects of employment fading in the long run. In contrast, Morlin and Pariboni (2024) and Serrano (2019) introduce the unemployment rate itself to determine workers' claims in their conflict inflation model. This assumption is in line with the typical assumption in short-term models, as, for instance, in the foundational work of Rowthorn (1977). Moreover, such an assumption allows for a lasting impact of unemployment on distribution, thus opening the door to an investigation of the political economy of autonomous demand (Morlin et al., 2024).

Besides the persistent effects of labour market conditions on income distribution, Morlin and Pariboni's (2024) model also provides a fruitful framework for exploring cyclical dynamics of growth and distribution—an aspect the authors themselves did not fully investigate. In the following sections, we will revisit the dynamic stability analysis of the SSM, extended to include income distribution. Our focus will be on identifying the conditions under which cyclical dynamics may emerge in this enhanced framework.

3. The stability analysis revisited

Morlin and Pariboni's (2024) model incorporates conflict inflation into an SSM, where income distribution and inflation are endogenous and influenced by the bargaining power of workers, which is mediated by the unemployment rate. In this section, we resume their analysis by adopting the traditional method of linearizing the system around the equilibrium point and calculating the eigenvalues of the Jacobian matrix. But here, instead of directly relying on the more widely used Routh-Hurwitz criteria, which provides conditions for stability in terms of the characteristic polynomial's coefficients, you have chosen to follow the stability criterion put forward by Asada and Yashida (2003).

The Asada and Yashida (2003) approach prepares the terrain for the use of the Hopf bifurcation theorem, which requires that the system have a pair of purely imaginary eigenvalues and no other eigenvalue with zero real parts. Additionally, this method allows for a more nuanced understanding of how oscillatory behaviour arises in a higher-dimensional system, which is more complex to handle using the standard Routh-Hurwitz criteria. By following Asada and Yashida (2003), we can more easily approach the system's dynamics to demonstrate how it can enter a regime of oscillations. Morlin and Pariboni (2023) model the evolution of four key variables: the investment share (h), capacity utilization (μ), the unemployment rate (u), and the wage share (ω) according to the following system of differential equations:

$$h' = \gamma(\mu - \mu_n) = f_1(h, \mu, u, \omega) \tag{1}$$

$$\mu' = \mu\left[g_Z + \sigma\phi\omega' + \sigma\gamma h(\mu - \mu_n) - \left(\frac{h\mu}{v}\right) + \delta\right] = f_2(h, \mu, u, \omega) \tag{2}$$

$$u' = (1 - u)[\beta_0 - \beta_1 u - g_Z - \sigma\phi\omega' - \sigma\gamma h(\mu - \mu_n)] = f_3(h, \mu, u, \omega) \tag{3}$$

$$\omega' = \frac{\omega\{(1-\alpha_1)\lambda_2\omega_k + \alpha_2(1-\lambda_1)(\theta_0 - \theta_1 u) - [(1-\alpha_1)\lambda_2 + \alpha_2(1-\lambda_1)]\omega\}}{1-\alpha_1\lambda_1} = f_4(h, \mu, u, \omega) \tag{4}$$

The first equation describes how the investment share evolves based on the difference between the actual capacity utilization (μ) and its normal level (μ_n), with γ representing the sensitivity of investment to deviations in capacity utilization. The second equation captures the dynamics of capacity utilization, influenced by the growth rate of autonomous demand (g_Z), the wage share's rate of change (ω'), and the interaction between investment share and capacity utilization, among other factors. The third equation explains the change in the unemployment rate (u), which is affected by factors such as the unemployment rate and parameters such as β_0 and β_1 and adjustments driven by changes in wage share and capacity utilization. Finally, the fourth equation governs the evolution of the wage share, which depends on the relative power of workers and firms in wage bargaining, as represented by parameters such as α_1, λ_1 and λ_2. The relevant equilibrium $P^*(h^*, \mu^*, u^*, \omega^*)$ is given by:

$$h^* = \frac{v(1+\delta)}{\mu_n} g_Z \tag{5}$$

$$\mu^* = \mu_n \tag{6}$$

$$u^* = \frac{\beta_0 - g_Z}{\beta_1} \tag{7}$$

$$\omega^* = \frac{(1-\alpha_1)\lambda_2\omega_k + \alpha_2(1-\lambda_1)\left[\Theta_0 - \frac{\Theta_1\beta_0 + g_Z\Theta_1}{\beta_1}\right]}{(1-\alpha_1)\lambda_2 + \alpha_2\,(1-\lambda_1)} \tag{8}$$

We can rewrite equation (8) as:

$$\omega^* = \frac{\beta_1\Phi_1 + \Phi_2(g_Z - \beta_0)}{\beta_1\Phi_3} \tag{8'}$$

where:

$$\Phi_1 = (1-\alpha_1)\lambda_2\omega_k + \alpha_2(1-\lambda_1)\Theta_0 \tag{9}$$

$$\Phi_2 = \alpha_2\,(1-\lambda_1)\Theta_1 \tag{10}$$

$$\Phi_3 = (1-\alpha_1)\lambda_2 + (1-\lambda_1)\alpha_2 \tag{11}$$

The linearization of the system around the equilibrium is assessed in terms of the Jacobian matrix, and its eigenvalues are calculated to ascertain the stability of the equilibrium. One possible approach to determine the asymptotic stability of the system is to analyze the characteristic equation of the Jacobian matrix, which can be expressed as follows:

$$\Delta\big(\lambda, J(P^*)\big) = \lambda^4 + S_1\lambda^3 + S_2\lambda^2 + S_3\lambda + S_4 = 0 \tag{12}$$

where S_1, S_2, S_3 and S_4 stand for the sums of minors of order one (trace), two, three and four (determinant) of the Jacobian matrix, respectively shown in the appendix. The Routh-Hurwitz criteria, necessary and sufficient conditions for all the roots of the characteristic polynomial to have negative real parts are R_1: $S_1 > 0, S_2 > 0, S_3 > 0, S_4 > 0$, and R_2: $\phi(P^*) = S_1S_2S_3 - S_3^2 - S_1^2S_4 > 0$. If the above conditions are satisfied then the non-zero equilibrium P^* will be locally and asymptotically stable. As the expressions for S_1, S_2, S_3 and S_4 involve cumbersome terms, we focus on an alternative route which consists of finding the roots of this characteristic equation (the eigenvalues) using Python. The four eigenvalues are given by:

$$\lambda_{1,2} = \frac{g_Z\gamma\sigma - \nu}{2g_Z} \pm \frac{\sqrt{\nu^2[1-4(1+\delta)g_Z^2\gamma\mu_n^2] + g_Z\gamma\sigma(g_Z\gamma\sigma - 2\nu)}}{2g_Z} \tag{17}$$

$$\lambda_{3,4} = \pm\sqrt{-\frac{\Theta_1\alpha_2\varphi\sigma(\lambda_1-1)(\alpha_1\lambda_1-1)}{(\alpha_1\lambda_2-1)(\beta_0+g_Z-1)}} \tag{18}$$

Some authors such as Araujo and Moreira (2023) and Nikiforos et al. (2023, p. 4) referring to the baseline system consider that eigenvalues are very likely to be complex, meaning that the baseline model has an equilibrium with convergent dumped cyclical trajectories. Indeed, in a 4-dimensional system, a possible way to ensure the asymptotic stability of the equilibrium with dumped oscillations is that the eigenvalues are complex with negative real parts. When the system has complex conjugate eigenvalues with negative real parts, the behaviour near the equilibrium point is oscillatory but the amplitude of the oscillations decays over time. Without complex eigenvalues, the system would not exhibit oscillatory behaviour, and if the real parts were positive, the system would become unstable.

Proposition 1:

If the following conditions are satisfied $g_Z\gamma\sigma < \nu$, $(1+\delta)(\,4g_Z^2\gamma\mu_n^2) > 1$ and $\lambda_1 < 1, \alpha_1\lambda_1 < 1$, $\alpha_1\lambda_2 < 1$ and $g_Z + \beta_0 < 1$, then the equilibrium $P^*(h^*, \mu^*, u^*, \omega^*)$ is asymptotic stable.

Proof.

To ensure asymptotic stability we have to guarantee that all these eigenvalues are complex and that the real part of $\lambda_{1,2}$ must be negative.The condition for the real part of $\lambda_{1,2}$ to be negative is that $g_Z\gamma\sigma < \nu$. Imposing this condition naturally implies that $g_Z\gamma\sigma < 2\nu$ and an additional sufficient condition for guaranteeing the negativity of the term within the root is $(1+\delta)(\,4g_Z^2\gamma\mu_n^2) > 1$. A possible combination of parameters that guarantee that $\lambda_{3,4}$ are complex roots are: $\lambda_1 < 1$, $\alpha_1\lambda_1 < 1, \alpha_1\lambda_2 < 1$ and $g_Z + \beta_0 < 1$. In this case, the equilibrium point is indeed a stable spiral. This means that trajectories near the equilibrium will spiral towards the equilibrium as time progresses. The system shows both oscillatory behaviour (due to the imaginary part of the eigenvalues) and convergence to the equilibrium (due to the negative real part of the eigenvalues). c.q.d.

An important relationship between the coefficients of the characteristic polynomial and the eigenvalues can be grasped from the following proposition, which is based on theorem 2 by Asada and Yashida (2003, p.527).

Proposition 2

If the polynomial equation has a pair of purely imaginary roots with negative real parts and real parts of the other two roots ($\lambda_{1,2}$) are not zero then $\phi = S_1 S_2 S_3 - S_3^2 - S_1^2 S_4 = 0$.

Proof. We know that:

$$S_1 = -\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \tag{P1}$$

$$S_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 \tag{P2}$$

$$S_3 = -\lambda_1\lambda_2\lambda_3 - \lambda_1\lambda_2\lambda_4 - \lambda_1\lambda_3\lambda_4 - \lambda_2\lambda_3\lambda_4 \tag{P3}$$

$$S_4 = \lambda_1\lambda_2\lambda_3\lambda_4 \tag{P4}$$

Assume that the characteristic equation (12) has a pair of purely imaginary roots, namely $\lambda_{3,4} = \pm\beta i$, where $\beta > 0$, and real parts of the other two roots ($\lambda_{1,2}$) are not zero. Substituting $\lambda_{3,4} = \pm\beta i$ into (P1), (P2), (P3) and (P4) it yields:

$$S_1 = -(\lambda_1 + \lambda_2) \tag{P5}$$

$$S_2 = \beta^2 + \lambda_1\lambda_2 \tag{P6}$$

$$S_3 = -(\lambda_1 + \lambda_2)\beta^2 = S_1\beta^2 \tag{P7}$$

$$S_4 = \beta^2\lambda_1\lambda_2 \tag{P8}$$

By inserting S_1, S_2, S_3 and S_4 into $\phi = S_1 S_2 S_3 - S_3^2 - S_1^2 S_4$, one obtains after some algebraic manipulation that $\phi = S_1^2\beta^2(\beta^2 + \lambda_1\lambda_2) - S_1^2\beta^2\,\lambda_1\lambda_2 - S_1^2\beta^4 = 0$ c.q.d.

We can also prove the following lemma, which will be useful in using the Hopf bifurcation theorem.

Lemma: If the polynomial equation has a pair of purely imaginary roots with negative real parts and real parts of the other two roots ($\lambda_{1,2}$) are negative then $S_1 > 0, S_3 > 0$ and $S_4 > 0$

Proof. Assume that the characteristic equation (12) has a pair of purely imaginary roots, namely $\lambda_{3,4} = \pm\beta i$, where $\beta > 0$, and real parts of the other two roots ($\lambda_{1,2}$) are negative. From (P5), the imaginary parts of $\lambda_{1,2}$ cancel out and as the real part is negative.

Hence $S_1 > 0$. From (P7), $S_3 = S_1\beta^2 > 0$, and from (P8), $S_4 = \beta^2\lambda_1\lambda_2 = \beta^2(\alpha'^2 + \beta'^2) > 0$, where α' and β' are the real and the imaginary part of λ_1 and λ_2. c.q.d.

To show that the Hopf-Andronov-Poincaré bifurcation may be performed on the system of equations (1) – (4) with bifurcation parameter g_Z, we shall make use of the following version of the Hopf bifurcation theorem by Asada and Yashida (2003, p. 526).

Theorem 1: Suppose that the dynamical system has the following properties $(H1) - (H3)$:

$(H1)$ This system has a smooth curve of equilibria $f(x_*(\mu); \mu) = 0$.

$(H2)$ The characteristic equation $|\lambda I - Df(x_*(\mu_o); \mu_o)| = 0$ has a pair of pure imaginary roots $\lambda(\mu_o), \overline{\lambda(\mu_o)}$ and no other root with zero real parts, where $Df(x_*(\mu_o); \mu_o)$ is the Jacobian matrix of this system at $(x_*(\mu_o); \mu_o)$ with the parameter value μ_o.

$(H3)$: $Re\left[\frac{d\lambda(\mu)}{d\mu}\right]_{\mu=\mu_o} \neq 0$, where Reλ(μ) is the real part of $\lambda(\mu)$.

Then, there exists a continuous function $\mu(\gamma)$ with $\mu(0) = \mu_o$, and for all sufficiently small values of $\gamma \neq 0$, there exists a continuous family of non-constant periodic solution $x(t, \gamma)$ for the above dynamical system, which collapses to the equilibrium point $x^*(t_o)$ as $\gamma \to 0$. The period of the cycle is close to $2\pi/Im\,\mu_o$, where $Im\,\mu_o$ is the imaginary part of $\lambda(\mu_o)$. This leads to the following theorem provides us very useful criterion for a complete mathematical characterization of the Hopf bifurcation in the four-dimensional system.

To use the Hopf bifurcation we need to choose a convenient parameter that makes the system satisfy conditions (H1) – (H3). Note that a suitable choice is the parameter g_Z insofar as h^*, u^* and, ω^* depend on it. Besides, we know that the $\lambda_3(g_Z)$ and $\lambda_4(g_Z)$ are a pair of pure imaginary eigenvalues which depend on g_Z, and the system has no other root with zero real parts since $\lambda_1(g_Z)$ and $\lambda_2(g_Z)$ are complex conjugate with a negative real part. Hence, what remains to be shown to prove that the system enters a Hopf bifurcation is the transversality condition (H3).

But theorem 3 of Asada and Yashida (2003, p.527) shows that we can replace the transversality condition $(H3)$, namely $\left[\frac{d\phi(g_Z)}{dg_Z}\right]_{g_Z=g_Z^*} \neq 0$, by the following set of conditions: $S_1(g_Z)S_3(g_Z) > 0$, $S_4(g_Z) \neq 0$ and $\phi(g_Z) = S_1(g_Z)S_2(g_Z)S_3(g_Z) - S_3^2(g_Z) - S_1^2S_4(g_Z) = 0$. Hence we can prove the following proposition:

Proposition. If the polynomial equation has a pair of purely imaginary roots with negative real parts and real parts of the other two roots ($\lambda_{1,2}$) are negative then the transversality condition holds.

Proof. We know from the lemma that $S_1(g_Z) > 0$ and $S_3(g_Z) > 0$, which allows us to conclude that $S_1(g_Z)S_3(g_Z) > 0$. Besides $S_4 > 0$, which also ensures that $S_4(g_Z) \neq 0$. Proposition 2 shows us that $\phi(g_Z) = 0$. Hence, from theorem 3 of Asada and Yashida (2003, p.527) we conclude that $\left[\frac{d\phi(g_Z)}{dg_Z}\right]_{g_Z=g_Z^*} \neq 0$. c.q.d.

We now present numerical simulations to verify the situation of a Hopf bifurcation for the system. Here, we consider $\gamma = 0.03326115367, \mu_n = 0.8, \sigma = 10, \phi = 0.1, \alpha_1 = 0.25$, $\alpha_2 = 0.75$, $\lambda_1 = 0.5, \lambda_2 = 0.9, \omega_K = 0.35, \Theta_0 = 0.5, \Theta_1 = 1.0, \upsilon = 3.0$, $\beta_0 = 0.5, \beta_1 = 1.0$ and $\delta = 0.05$. The bifurcation parameter:
$g_Z \in [g_Z^1, g_Z^2]$ =[0.030000008865, 0.030000008865000000001],

We shall prove that $\phi(g_Z^1) < 0$ and $\phi(g_Z^2) > 0$. Therefore, if we increase the values of g_Z in this interval, by the Intermediate Value Theorem, applied to function $\phi(g_Z)$, there is at least one $g_Z^* \epsilon[\ \in [g_Z^1, g_Z^2]$ such that $\phi(g_Z^*) = 0$, that is, the characteristic equation $|\lambda I - Df(x_*(g_Z); g_Z)| = 0$ has a pair of pure imaginary roots $\lambda(g_Z), \overline{\lambda(g_Z)}$ and no other root with zero real parts, where $Df(x_*(g_Z); g_Z)$ is the Jacobian matrix of this system at $(x_*(g_Z);)$ with the parameter value $g_Z = g_Z^*$. So it is clear that the system enters into a Hopf bifurcation when g_Z increases and reaches a bifurcation point. Hence, $h(t), \mu(t), u(t)$ and $\omega(t)$ show oscillations when $g_Z = g_Z^*$.

When $g_Z^1 = 0.0300000088$, the system has a positive equilibrium $P_1^* = (0.3000000330, 0.8, 0.02999999120, 0.3928571460)$ which is a stable equilibrium, with eigenvalues $\lambda_{1,2} = -7.45774969557189\ 10^{-13} \pm 0.0468016689802291\ i$, and

$\lambda_{3,4} = -0.639142742949255 \pm 0.189486112198758\ i$. Here the Jacobian is

$$J(P_1^*) = \begin{bmatrix} 0 & 0.009978347199 & 0 & 0 \\ -0.21333333334 & -0.00017323120 & -0.1346938786 & -0.3771428602 \\ 0 & -0.09678996872 & -0.8066836795 & 0.4572857221 \\ 0 & 0 & -0.1683673483 & -0.4714285752 \end{bmatrix},$$

The coefficients of the characteristic polynomial are $S_1 = 1.278285486, S_2 = 0.4465988289, S_3 = 0.002799951696$, and $S_4 = 0.0009734305505$. Besides, $\phi(g_Z^1) = b_1 b_2 b_3 - b_4 b_1^2 - b_3^2 = 5.00\ 10^{-13} > 0$.

When $g_Z^2 = 0.030000008865000000001$, the system has a positive equilibrium $P_1^* = (0.3000000333, 0.8, 0.02999999113, \ 0.3928571460)$, which is an unstable spiral, with eigenvalues: $\lambda_{1,2} = 4.56878979093744\ 10^{-12} \pm 0.0468016690042829\ i$, and $\lambda_{3,4} = -0.639142742999569 \pm 0.189486112119760\ i$. The Jacobin is:

$$J(P_1^*) = \begin{bmatrix} 0 & 0.009978347209 & 0 & 0 \\ -0.21333333334 & -0.00017323119 & -0.1346938786 & -0.3771428602 \\ 0 & -0.09678996882 & -0.8066836796 & 0.4572857221 \\ 0 & 0 & -0.1683673483 & -0.4714285752 \end{bmatrix},$$

The coefficients of the characteristic polynomial are $S_1 = 1.278285486, S_2 = 0.4465988289, S_3 = 0.002799951692$ and $S_4 = 0.0009734305505$. Besides, $\phi(g_Z^2) = b_1 b_2 b_3 - b_4 b_1^2 - b_3^2 = -1.478\ 10^{-12}$. This means that a simple Hopf bifurcation occurs at $g_Z = g_Z^*$, when $g_Z^* = 0.030000008865$. The system has a positive equilibrium, namely $P_1^* = (0.3000000332, 0.8, 0.02999999114, 0.3928571460)$, which is a stable equilibrium, with eigenvalues: $\lambda_{1,2} \cong \ 0 \pm 0.0468016690043024\ i$, and

$\lambda_{3,4} = -0.639142742999426 \pm 0.189486112119269\ i$.

Here the Jacobian is given by:

$$J(P_1^*) = \begin{bmatrix} 0 & 0.009978347209 & 0 & 0 \\ -0.21333333334 & -0.00017323120 & -0.1346938786 & -0.3771428602 \\ 0 & -0.09678996882 & -0.8066836796 & 0.4572857221 \\ 0 & 0 & -0.1683673483 & -0.4714285752 \end{bmatrix}$$

And the coefficients of the characteristic polynomial are positive: $S_1 = 1.278285486, S_2 = 0.4465988289, S_3 = 0.002799951698, S_4 = 0.0009734305516$. Besides: $\phi(g_Z^*) = S_1 S_2 S_3 - S_4 S_1^2 - S_3^2 = -5.11\ 10^{-13} \cong 0$

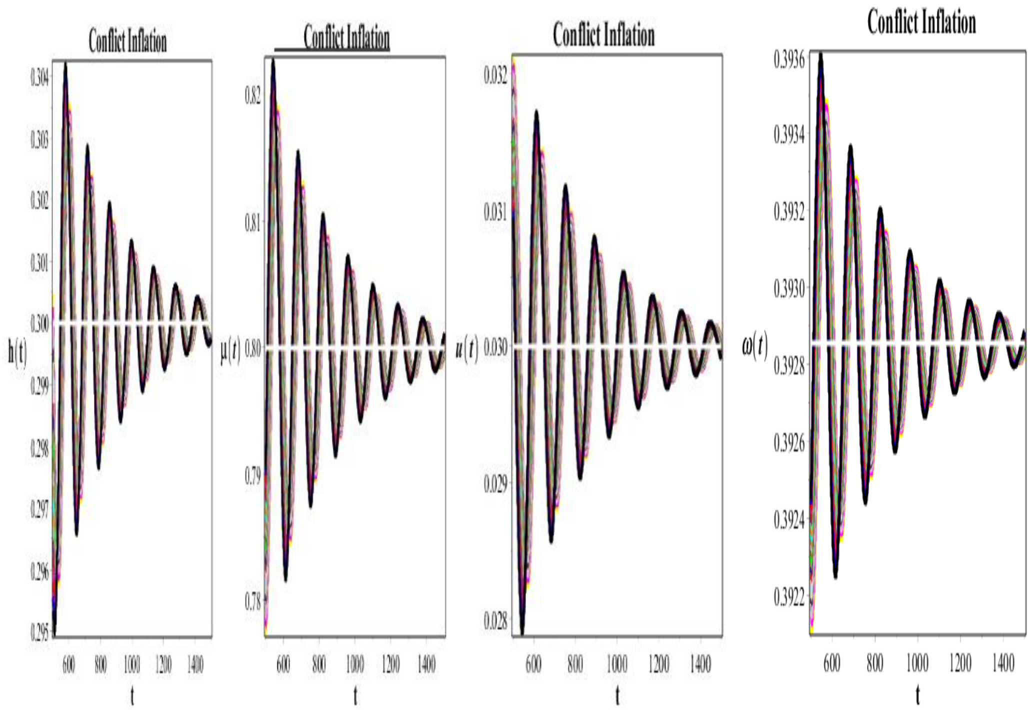
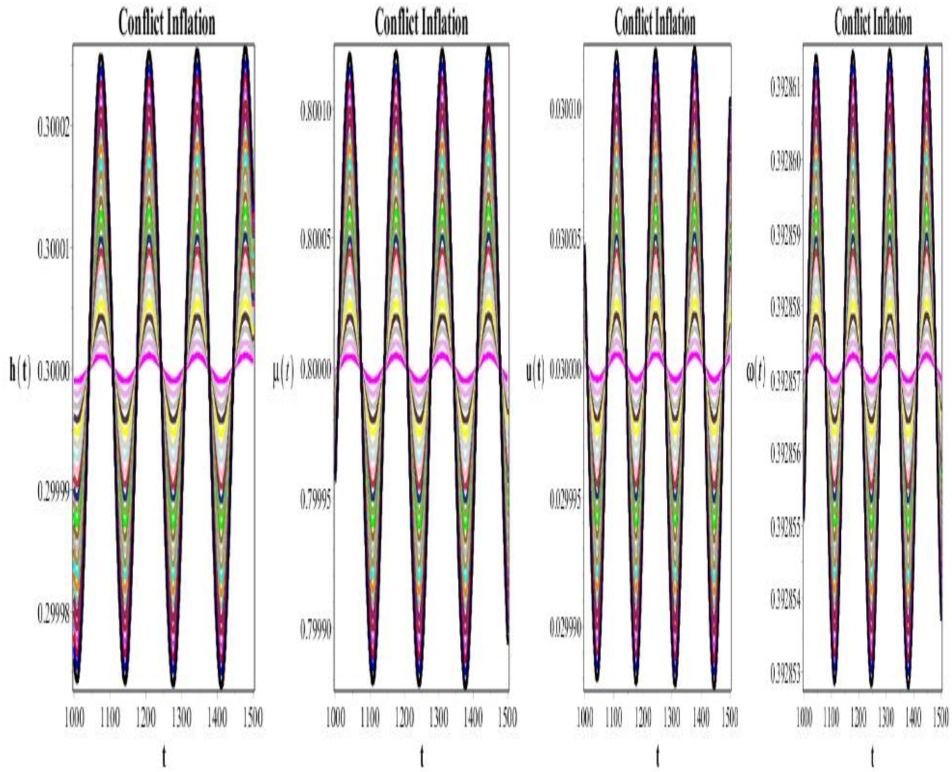
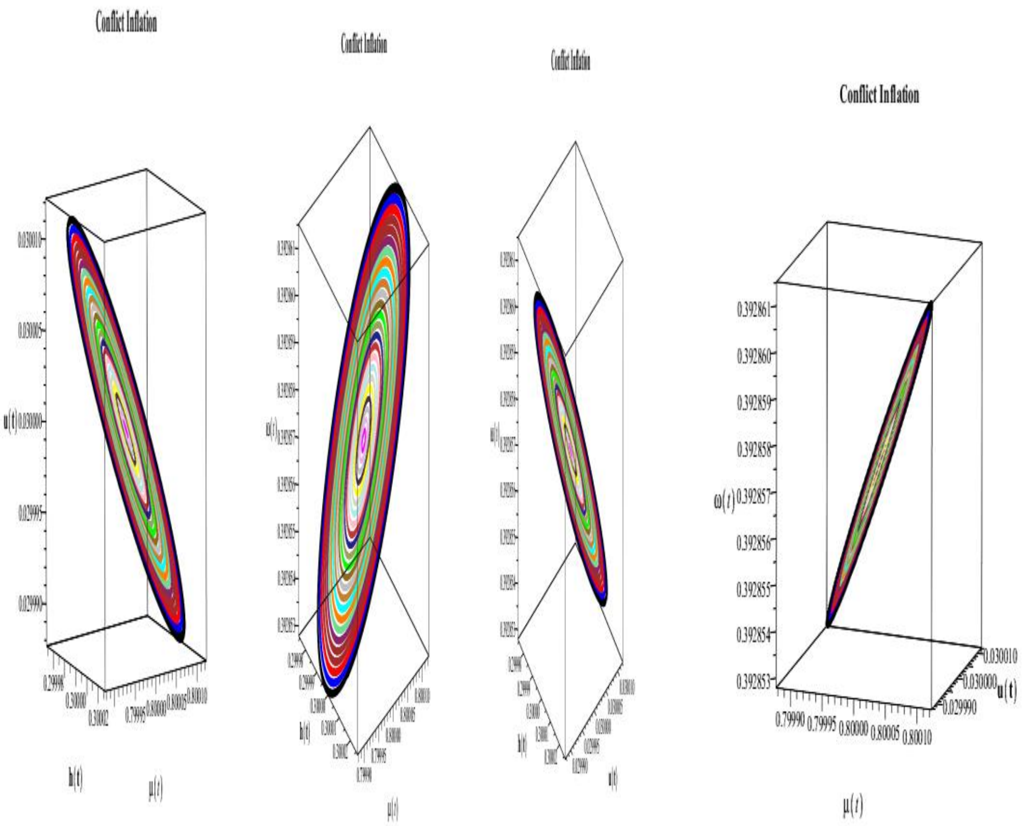


Figure 1. Time series of h, μ, u and ω versus t for twenty-five initial conditions $(h(0), \mu(0), u(0), \omega(0))$ $(see\ Appendix\ A)$., where $t = 500 \ldots 1500$. The steady state $P^* = (0.3, 0.8, 0.03, 0.3928571429$ is locally asymptotically stable.





5. Concluding Remarks

This study established conditions for the existence of limit cycles of growth, unemployment, and wage share within the SSM with endogenous income distribution. We revisited the mathematical structure of the model proposed by Morlin and Pariboni (2024) to refine its stability analysis. By applying a robust stability criterion, we showed that under certain conditions, the model admits limit cycles, leading to sustained oscillations. Our findings demonstrate that persistent endogenous cycles are not only consistent with the SSM but can be derived directly from its theoretical framework. This insight significantly enhances our understanding of macroeconomic fluctuations within the SSM paradigm, providing a robust theoretical foundation for cyclical patterns that align with a demand-led growth perspective. By integrating endogenous distribution dynamics, this research offers a more comprehensive analytical framework for studying economic cycles in autonomous demand-led growth models. This result also enhances the theoretical foundations of the SSM with endogenous distribution, offering a new perspective on how conflict inflation and autonomous demand can interact to produce cyclical economic dynamics.

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Appendix

The Jacobian matrix due to the linearization of the system about the relevant equilibrium $P^*(h^*, \mu^*, u^*, \omega^*) \epsilon R_+^4$ is given by:

$$J(P^*) = Df(g_Z) = \begin{bmatrix} f_{1h} f_{1\mu} f_{1u} f_{1\omega} \\ f_{2h} f_{2\mu} f_{2u} f_{2\omega} \\ f_{3h} f_{3\mu} f_{3u} f_{3\omega} \\ f_{4h} f_{4\mu} f_{4u} f_{4\omega} \end{bmatrix}$$

where $f_{ij} = \frac{\partial \mathrm{f_i}}{\partial \mathrm{x_j}}(P^*)$,$1 \leq i, j \leq 4, x_j = h, \mu, u, \omega,$ that is:

$f_{1h}(g_Z) = 0$

$f_{1\mu}(g_Z) = \gamma v g_Z \frac{(1+\delta)}{\mu_n}$

$f_{1u}(g_Z) = 0$

$f_{1\omega}(g_Z) = 0$

$$f_{2h}(g_Z) = -\frac{(\mu_n)^2}{v}$$

$$f_{2\mu}(g_Z) = \left(\gamma\sigma - \frac{1}{v}\right) v g_Z$$

$$f_{2u}(g_Z) = -\frac{\sigma\phi\Phi_2\mu_n[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)]}{(1-\alpha_1\lambda_1)\beta_1\Phi_3}$$

$$f_{2\omega}(g_Z) = -\frac{\sigma\phi\mu_n[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1}$$

$$f_{3h}(g_Z) = 0$$

$$f_{3\mu}(g_Z) = -\gamma\sigma\left(1-\frac{\beta_0-g_Z}{\beta_1}\right)\frac{v(1+\delta)}{\mu_n} g_Z$$

$$f_{3u}(g_Z) = \left\{\frac{\sigma\phi\Phi_2[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1\,\Phi_3} - \beta_1\right\}\left(1-\frac{\beta_0-g_Z}{\beta_1}\right)$$

$$f_{3\omega}(g_Z) = \frac{\sigma\phi[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1}\left(1-\frac{\beta_0-g_Z}{\beta_1}\right)$$

$$f_{4h}(g_Z) = 0$$

$$f_{4\mu}(g_Z) = 0$$

$$f_{4u}(g_Z) = -\frac{\Phi_2[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1\Phi_3}$$

$$f_{4\omega}(g_Z) = -\frac{\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1}{(1-\alpha_1\lambda_1)\beta_1}$$

The coefficients of the characteristic polinomial evaluated in the relevant equilibrium point P^*, and are given by:

$$S_1(g_Z) = -\left(\gamma\sigma-\frac{1}{v}\right)vg_Z - \left\{\frac{\sigma\phi\Phi_2[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1\,\Phi_3} - \beta_1\right\}\left(1-\frac{\beta_0-g_Z}{\beta_1}\right) + \frac{\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1}{(1-\alpha_1\lambda_1)\beta_1}$$

$$S_2(g_Z) = -\gamma g_Z\mu_n - \left(\gamma\sigma-\frac{1}{v}\right)vg_Z\frac{\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1}{(1-\alpha_1\lambda_1)\beta_1} + \left(1-\frac{\beta_0-g_Z}{\beta_1}\right)\left\{\left(\gamma\sigma - \frac{1}{v}\right)vg_Z\left[\frac{\sigma\phi\Phi_2[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1\,\Phi_3} - \beta_1\right] - \frac{\sigma\phi\Phi_2[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)]\gamma\sigma v(1+\delta)g_Z}{(1-\alpha_1\lambda_1)\beta_1\Phi_3} - \left[\frac{\sigma\phi\Phi_2[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1\,\Phi_3} - \beta_1\right]\frac{\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1}{(1-\alpha_1\lambda_1)\beta_1} - \frac{\sigma^2\phi\Phi_2 v(1+\delta)\gamma g_Z[\beta_1\Phi_1+\Phi_2(g_Z-\beta_0)\Theta_1]}{(1-\alpha_1\lambda_1)\beta_1\Phi_3}\left(1-\frac{\beta_0-g_Z}{\beta_1}\right)\right\}$$

$$S_3 = \frac{g_Z\,\gamma\mu_n}{\Phi_3\beta_1^3(\alpha_1^2\,\lambda_1^2 - 2\,\alpha_1\lambda_1 + 1)}[-\Phi_1{}^2\Phi_2\beta_0\beta_1{}^2\varphi\sigma + \Phi_1{}^2\Phi_2\beta_1{}^2 g_Z\varphi\sigma$$

$$+ \Phi_1\,\Phi_2{}^2\Theta_1\,\beta_0{}^2\beta_1\varphi\sigma + \Phi_1\Phi_2{}^2\Theta_1\beta_0\beta_1{}^2\varphi\sigma - \Phi_1\Phi_2{}^2\Theta_1\beta_1{}^2 g_Z\varphi\sigma$$

$$+ \Phi_1\Phi_2{}^2\Theta_1\beta_1 g_Z{}^2\varphi\sigma + \Phi_1\Phi_2{}^2\beta_0{}^2\beta_1\varphi\sigma - \Phi_1\Phi_2{}^2\beta_0\beta_1{}^2\varphi\sigma$$

$$+ \Phi_1\Phi_2{}^2\beta_1{}^2 g_Z\varphi\sigma + \Phi_1\Phi_2{}^2\,\beta_1\,g_Z{}^2\varphi\sigma - \Phi_1\Phi_3\alpha_1\beta_1{}^4\lambda_1 + \Phi_1\Phi_3\beta_1{}^3 g_Z$$

$$- \Phi_2{}^3\Theta_1{}^2\beta_0{}^2\beta_1\,\varphi\,\sigma + 2\Phi_2{}^3\Theta_1{}^2\beta_0\,\beta_1 g_Z\varphi\sigma - \Phi_2{}^3\Theta_1{}^2\beta_1 g_Z{}^2\varphi\sigma$$

$$+ 3\Phi_2{}^3\Theta_1\beta_0{}^2 g_Z\varphi\sigma - 3\Phi_2{}^3\Theta_1\beta_0 g_Z{}^2\varphi\sigma + \Phi_2{}^3\Theta_1 g_Z{}^3\varphi\sigma\ ]$$

$$S_4 = -\frac{g_Z\gamma\mu_n}{\Phi_3\beta_1^3(\alpha_1^2\,\lambda_1^2 - 2\,\alpha_1\lambda_1 + 1)}[-\Phi_1^2\Phi_2\beta_0\beta_1^2\varphi\sigma + \Phi_1^2\Phi_2\beta_1^2 g_Z\varphi\sigma$$

$$+ 2\Phi_1\Phi_2^2\Theta_1\beta_0^2\beta_1\varphi\sigma - 4\Phi_1\Phi_2^2\Theta_1\beta_0\beta_1 g_Z\varphi\sigma + 2\Phi_1\Phi_2^2\Theta_1\beta_1 g_Z^2\varphi\sigma$$

$$+ \Phi_1\Phi_3\alpha_1\beta_0\beta_1^3\lambda_1 - \Phi_1\Phi_3\alpha_1\beta_1^4\lambda_1 - \Phi_1\Phi_3\alpha_1\beta_1^3 g_Z\lambda_1 - \Phi_1\Phi_3\beta_0\beta_1^3$$

$$+ \Phi_1\Phi_3\beta_1^4 + \Phi_1\Phi_3\beta_1^3 g_Z - \Phi_2^3\Theta_1^2\beta_0^3\varphi\sigma + 3\Phi_2^3\Theta_1^2\beta_0^2 g_Z\varphi\sigma$$

$$- 3\Phi_2^3\Theta_1^2\beta_0 g_Z^2\varphi\sigma + \Phi_2^3\Phi_1^2 g_Z^3\varphi\sigma - \Phi_2\Phi_3\Theta_1\alpha_1\beta_0^2\beta_1^2\lambda_1$$

$$+ \Phi_2\Phi_3\Theta_1\alpha_1\beta_0\beta_1^3\lambda_1 + 2\Phi_2\Phi_3\Theta_1\alpha_1\beta_0\beta_1^2 g_Z\lambda_1 - \Phi_2\Phi_3\Theta_1\alpha_1\beta_1^3 g_Z\lambda_1$$

$$- \Phi_2\Phi_3\Theta_1\alpha_1\beta_1^2 g_Z^2\lambda_1 + \Phi_2\Phi_3\Theta_1\beta_0^2\beta_1^2 - \Phi_2\Phi_3\Theta_1\beta_0\beta_1^3$$

$$- 2\Phi_2\Phi_3\Theta_1\beta_0\beta_1^2 g_Z + \Phi_2\Phi_3\Theta_1\beta_1^3 g_Z + \Phi_2\Phi_3\Theta_1\beta_1^2 g_Z^2]$$